On the product of distributions with coincident point singularities *

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In this paper we generalize the result obtained by Gonzáles Domínguez, Scarfiello and Fisher (A. Gonzáles Domínguez and R. Scarfiello, Rev. Un. Mat. Argentina, Volumen de Homenaje a Beppo Levi (1956) 58–67; B. Fisher, Proc. Cambridge Philos. Soc. 72 (1972) 201–204). This result can be used in quantum field theory for the evaluation of products of propagators of the fields. With this new result we obtain the product $(c^2 + P)^{-n} \cdot \delta^{m-1}(c^2 + P)$. As a physical example, we evaluate the self-energy Green function of a massless scalar field.

1. Introduction

Our purpose is to evaluate the product $x^{-n} \cdot \delta^{m-1}(x)$ $(n, m = 1, 2, \ldots)$, which appears in applications, essencially in the quantum theory of fields, in the evaluation of products of propagators.

The distribution $\delta^{(s)}(x)$ is the sth derivative of the $\delta(x)$ distribution and x^{-n} $(n = 1, 2, \ldots)$ is the distribution defined by the formula (see [8])

$$x^{-n} = x_+^{-n} + (-1)^n x_-^n, \tag{1}$$

where (see [11, pp. 86, 89])

$$\left(x_+^{-n}, \varphi\right) = \int_0^\infty x^{-n} \left[\varphi(x) - \varphi(0) - x\varphi'(0) - \cdots - \frac{x^{(n-1)}}{(n-1)!}\varphi^{(n-1)}(0)H(1-x)\right] \mathrm{d}x, \tag{2}$$

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$$\left(x_-^{-n},\varphi\right)=\int_0^\infty x^{-n}\Bigg[\varphi(-x)-\varphi(0)+x\varphi'(0)+\cdots$$
$$+(-1)^{n-1}\frac{x^{(n-1)}}{(n-1)!}\varphi^{(n-1)}(0)H(1-x)\Bigg]\,\mathrm{d}x. \tag{3}$$

In (2) and (3), $H(x)$ is Heaviside's step function.

The distribution x^{-n} is sometimes denoted by *fp* x^{-n}, where the letters *fp* mean "finite part", for $n>1$; x^{-1} is usually denoted by *Pv* x^{-1}, where the letters *Pv* mean "principal value".

To obtain the product $x^{-n}\cdot\delta^{(m-1)}(x)$ we also need the following formulae (see [11, pp. 93, 94]):

$$(x\pm \mathrm{i}0)^{-n}=x^{-n}\mp\pi\mathrm{i}\frac{(-1)^{n-1}}{(n-1)!}\delta^{(n-1)}(x), \tag{4}$$

$$(x\pm \mathrm{i}0)^{\lambda}=x_+^\lambda+\mathrm{e}^{\pm\lambda\pi\mathrm{i}}x_-^\lambda, \tag{5}$$

$$x_+^\lambda=\frac{(-1)^{n-1}\delta^{(n-1)}(x)}{(n-1)!(\lambda+n)}+F_{-n}(x_+,\lambda), \tag{6}$$

$$x_-^\lambda=\frac{\delta^{(n-1)}(x)}{(n-1)!(\lambda+n)}+F_{-n}(x_-,\lambda), \tag{7}$$

where $F_{-n}(x_+,\lambda)$ and $F_{-n}(x_-,\lambda)$ are the regular parts of the Laurent expansions (see [11, pp. 86, 88]) and $x_\pm^\lambda$ are the generalized functions defined by

$$x_+^\lambda=\begin{cases}x^\lambda & \text{for } x>0,\\ 0 & \text{for } x\leqslant 0,\end{cases} \tag{8}$$

and

$$x_-^\lambda=\begin{cases}|x|^\lambda & \text{for } x<0,\\ 0 & \text{for } x\geqslant 0.\end{cases} \tag{9}$$

Also we know that the following formulae are valid [10]:

$$\Gamma(z+n)=z(z+1)\cdots(z+n-1)\Gamma(z), \tag{10}$$

$$\frac{\Gamma(z)}{\Gamma(z-n)}=(-1)^n\frac{\Gamma(-z+n+1)}{\Gamma(-z+1)}, \tag{11}$$

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right)=\pi\sec(\pi z), \tag{12}$$

$$\Gamma(2z)=2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{13}$$

The latter is also known as the "duplication formulae" for Euler's Γ-function. In order to deal with multidimensional generalizations of all products evaluated in sections 2 and 3 we introduce the following definition:

Let $\phi(s)$ be a distribution of one variable s and let $\mathcal{U}(x_1, x_2, \ldots, x_n) \in C^\infty(R^n)$ be such that the $(n-1)$-dimensional manifold $\mathcal{U}(x_1, x_2, \ldots, x_n) = 0$ has no critical points; $\phi_{\mathcal{U}}(x)$ denotes the distribution defined by the formula (called the Leray formula [15])

$$\bigl\langle \phi_{\mathcal{U}}\bigl(x, \varphi(x)\bigr)\bigr\rangle = \bigl\langle \phi(s), \psi(s)\bigr\rangle, \tag{14}$$

where

$$\psi(s) = \int_{\mathcal{U}(x)=s} \varphi(x) W_{\mathcal{U}}(x, \mathrm{d}x). \tag{15}$$

Here $W_{\mathcal{U}}$ is an $(n-1)$-dimensional form on $\mathcal{U}$ defined as follows:

$$\mathrm{d}\mathcal{U} \wedge W_{\mathcal{U}} = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n,$$

and the orientation of the manifold $\mathcal{U}(x_1, x_2, \ldots, x_n) = s$ is such that

$$\det W_{\mathcal{U}}(x, \mathrm{d}x) > 0.$$

2. The multiplicative product $x^{-n} \cdot \delta^{(m-1)}(x)$

In this section we shall obtain the formula

$$x_{-n} \cdot \delta^{(m-1)}(x) = \frac{(-1)^n (m-1)!}{2(m+n-1)!} \delta^{(m+n-1)}(x), \quad n, m \in \mathcal{N},\ n, m \geqslant 1. \tag{16}$$

Here $\mathcal{N} =$ natural numbers.

We shall study this product for n and m positive integers taking into account the following three cases:

(1) $m = n$,

(2) $m > n$,

(3) $m < n$.

Formula (16) has been proved for $m = n$ by Gonzáles Domínguez and Scarfiello in [9]. It was later rediscovered by other authors (cf. [7,14]).

For $m = n = 1$ we obtain the well-known result [9]

$$x^{-1} \cdot \delta(x) = -\frac{1}{2}\delta'(x). \tag{17}$$

According to [11, pp. 96, 97] we can write

$$x_+^\lambda + \mathrm{e}^{\mathrm{i}\pi\lambda} x_-^\lambda = \left[x^{-n} + \frac{\pi \mathrm{i}(-1)^n}{(n-1)!}\delta^{(n-1)}(x)\right] + (\lambda + n)\left[\pi\mathrm{i}(-1)^n x_-^{-n} + (-1)^{n-1}\frac{\pi^2}{2}\,\frac{\delta^{(n-1)(x)}}{(n-1)!} + x^{-n}\ln|x|\right]$$

$$+\frac{(\lambda+n)^2}{2}\Bigg[(-1)^{n-1}\,\mathrm{i}\frac{\pi^2}{3}\,\frac{\delta^{(n-1)(x)}}{(n-1)!}+(-1)^{n-1}\pi^2 x_-^{-n} + 2\mathrm{i}\pi(-1)^n x_-^{-n}\ln x_- + x^{-n}\ln^2|x|\Bigg]. \tag{18}$$

Therefore from (18) we have

$$\lim_{\lambda\to -n}(x+\mathrm{i}0)^\lambda = \lim_{\lambda\to -n}\big(x_+^\lambda + \mathrm{e}^{\mathrm{i}\pi\lambda}x_-^\lambda\big) = x^{-n} + \frac{\pi\mathrm{i}(-1)^n}{(n-1)!}\delta^{(n-1)}(x). \tag{19}$$

From (19) and (2) we conclude that

$$\lim_{\lambda\to -n}(x\pm \mathrm{i}0)^\lambda = (x\pm\mathrm{i}0)^{-n}. \tag{20}$$

On the other hand, according to [11, p. 57],

$$\lim_{\lambda\to -n}\frac{x_+^\lambda}{\Gamma(\lambda+1)} = \delta^{(n-1)}(x), \quad n=1,2,\ldots, \tag{21}$$

where x_+^λ is given by (8).

Case 1: The product $x^{-n}\cdot\delta^{(n-1)}(x)$

The product $x^{-n}\cdot\delta^{(n-1)}(x)$ was presented in [9] and, more recently, in [7].

We now obtain (16) for $m=n$ using (19), (13) and the formulae of [4,5,16]. From (4), (5) and considering (20) we have

$$x^{-n} = \frac{1}{2}\big((x+\mathrm{i}0)^{-n} + (x-\mathrm{i}0)^{-n}\big) = \lim_{\lambda\to -n}\frac{1}{2}\big((x+\mathrm{i}0)^\lambda + (x-\mathrm{i}0)^\lambda\big). \tag{22}$$

Also, the following formulae are valid (see [5]):

$$x_\pm^\lambda\cdot x_\pm^\mu = x_\pm^{\lambda+\mu} \tag{23}$$

and

$$x_+^\lambda\cdot x_-^\mu = 0, \tag{24}$$

where λ and μ are complex numbers such that λ, μ and $\lambda+\mu\neq -k$, $k=1,2,\ldots$. Formulae (21) and (22) allow us to represent $\delta^{(m)}$ and x^{-n} as "canonically regularized" by these equations. In this way the canonically regularized form of the product we are looking for can be defined as the product of the corresponding λ-dependent expressions:

$$\begin{aligned} x^{-n}\cdot\delta^{(n-1)}(x) &= \lim_{\lambda\to -n}\frac{1}{2}\big((x+\mathrm{i}0)^\lambda + (x-\mathrm{i}0)^\lambda\big)\cdot\frac{x_+^\lambda}{\Gamma(\lambda+1)} \\ &= \lim_{\lambda\to -n}\big[x_+^\lambda + \cos\pi\lambda\, x_-^\lambda\big]\cdot\frac{x_+^\lambda}{\Gamma(\lambda+1)} \\ &= \lim_{\lambda\to -n}\bigg[\frac{x_+^\lambda x_+^\lambda}{\Gamma(\lambda+1)} + \cos\pi\lambda\,\frac{x_-^\lambda x_+^\lambda}{\Gamma(\lambda+1)}\bigg]. \end{aligned} \tag{25}$$

From (23)–(25) we get

$$x^{-n} \cdot \delta^{(n-1)}(x) = \lim_{\lambda \to -n} \frac{x_+^{2\lambda}}{\Gamma(\lambda+1)}. \tag{26}$$

By using the duplication formula (13) and taking the limit, we obtain

$$x^{-n} \cdot \delta^{(n-1)}(x) = \frac{(-1)^n}{2} \frac{(n-1)!}{(2n-1)!} \delta^{(2n-1)}(x). \tag{27}$$

Formula (27) coincides with the result of [7,9].

Case 2: The product $x^{-n} \cdot \delta^{(n-1)}(x)$ for $m > n$

If $m > n$, there exists $p \in \mathcal{N}$ such that $m = n + p$. Therefore

$$\lim_{\mu \to -m} \frac{x_+^{\mu}}{\Gamma(\mu+1)} = \lim_{\gamma \to -n} \frac{x_+^{\gamma-p}}{\Gamma(\gamma+1-p)}. \tag{28}$$

Then, from (21), (22) and (28) we have

$$\begin{aligned} x^{-n} \cdot \delta^{(m-1)}(x) &= \lim_{\gamma \to -n} \frac{1}{2}\left[(x+\mathrm{i}0)^{\gamma} + (x-\mathrm{i}0)^{\gamma}\right] \frac{x_+^{\gamma-p}}{\Gamma(\gamma+1-p)} \\ &= \lim_{\gamma \to -n}\left[x_+^{\gamma} + \cos\pi\gamma\, x_-^{\gamma}\right] \frac{x_+^{\gamma-p}}{\Gamma(\gamma+1-p)}. \end{aligned} \tag{29}$$

From (23), (24), (28) and (29) we conclude that

$$x^{-n} \cdot \delta^{(m-1)}(x) = \lim_{\gamma \to -n} \frac{x_+^{2\gamma-p}}{\Gamma(\gamma+1-p)} \tag{30}$$

and, by taking the limit, we conclude that

$$x^{-n} \cdot \delta^{(m-1)}(x) = \frac{(-1)^n}{2} \frac{(m-1)!}{(m+n-1)!} \delta^{(m+n-1)}(x), \quad m > n. \tag{31}$$

Case 3: The product $x^{-n} \cdot \delta^{(n-1)}(x)$ for $m < n$

If $m < n$, there exists $p \in \mathcal{N}$ such that $n = m + p$. Then, from (39) and taking into account that

$$\lim_{\mu \to -m} \frac{x_+^{\mu}}{\Gamma(\mu+1)} = \lim_{\gamma \to -n} \frac{x_+^{\gamma+p}}{\Gamma(\gamma+1+p)}, \tag{32}$$

we have

$$x^{-n} \cdot \delta^{(m-1)}(x) = \lim_{\gamma \to -n}\left[x_+^{\gamma} + \cos\pi\gamma\, x_-^{\gamma}\right] \frac{x_+^{\gamma+p}}{\Gamma(\gamma+1+p)}. \tag{33}$$

From (23) and (24) we obtain for (33)

$$x^{-n} \cdot \delta^{(m-1)}(x) = \lim_{\gamma \to -n} \frac{x_+^{2\gamma+p}}{\Gamma(\gamma+1+p)}. \tag{34}$$

And, by taking the limit, we finally get

$$x^{-n} \cdot \delta^{(m-1)}(x) = \frac{(-1)^n}{2} \frac{(m-1)!}{(m+n-1)!} \delta^{(m+n-1)}(x) \tag{35}$$

for all $m, n \geqslant 1$. This proves our assertion (16). It is clear that with a similar procedure we can prove that

$$x^{-n} \cdot x^{-m} = x^{-n-m} \tag{36}$$

and also that

$$\delta^{(m)}(x) \cdot \delta^{(n)}(x) = 0. \tag{37}$$

In fact, the last equation can be written as (cf. (28), $m = n + p$, $p \geqslant 0$)

$$\lim_{\lambda \to -n} \frac{x_+^{\lambda-p}}{\Gamma(\lambda+1-p)} \cdot \frac{x_+^{\lambda}}{\Gamma(\lambda+1)} = \lim_{\lambda \to -n} \frac{x_+^{2\lambda-p}}{\Gamma(\lambda+1-p)\Gamma(\lambda+n)} = 0.$$

The numerator has a simple pole, while the denominator has a double pole.

The last equation leads to the following theorem [4,5]:

Theorem. The product of two distributions with point support is zero.

3. The multiplicative product $(c^2 + P)^{-n} \cdot \delta^{(m-1)}(P)$

In this section we want to extend result (54) to certain kinds of n-dimensional distributions.

We will start with some definitions.

Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n-dimensional Euclidean space $\mathcal{R}^n$. Consider a non-degenerate quadratic form in n variables of the form

$$P \equiv P(x) = x_1^2 + x_2^2 + \cdots + x_\mu^2 - x_{\mu+1}^2 - x_{\mu+2}^2 - \cdots - x_{\mu+\nu}^2, \tag{38}$$

where $\mu + \nu = n =$ dimension of the space. The distributions $(c^2 + P \pm \mathrm{i}0)^\lambda$ are defined by (see [11, p. 289])

$$\left(c^2 + P \pm \mathrm{i}0\right)^\lambda = \lim_{\varepsilon \to 0} \left(c^2 + P \pm \mathrm{i}\varepsilon |x|^2\right)^\lambda, \tag{39}$$

where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$, $\varepsilon > 0$, λ is a complex number and c^2 is a positive real number.

It is useful to state an equivalent definition of the distributions $(c^2 + P \pm \mathrm{i}0)^\lambda$. This definition involves the distributions

$$\left(c^2+P\right)_+^\lambda = \begin{cases} (c^2+P)^\lambda & \text{for } c^2+P>0, \\ 0 & \text{for } c^2+P \leqslant 0, \end{cases} \tag{40}$$

and

$$\left(c^2+P\right)_-^\lambda = \begin{cases} (-c^2-P)^\lambda & \text{for } c^2+P<0, \\ 0 & \text{for } c^2+P \geqslant 0. \end{cases} \tag{41}$$

From [6] we have

$$\left(c^2+P\pm \mathrm{i}0\right)^\lambda = \left(c^2+P\right)_+^\lambda + \mathrm{e}^{\pm \mathrm{i}\pi\lambda}\left(c^2+P\right)_-^\lambda, \tag{42}$$

and from this formula we conclude inmediately that

$$\left(c^2+P\pm \mathrm{i}0\right)^\lambda = \left(c^2+P\right)^\lambda \tag{43}$$

when λ is a positive integer.

It can be proved [6, p. 573, formula (2.14), p. 575, formula (3.5)] that

$$\left(c^2+P\pm \mathrm{i}0\right)^{-k} = \left(c^2+P\right)^{-k} \mp \frac{(-1)^{k-1}\pi \mathrm{i}}{(k-1)!}\delta^{(k-1)}\left(c^2+P\right), \tag{44}$$

where $(c^2 + P)^{-k}$ is the regular part of the Laurent expansion of $(c^2 + P)_+^\lambda$ near $\lambda = -k$, namely,

$$\left(c^2+P\right)^{-k} = \lim_{\lambda\to -k}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[(\lambda+k)\left(c^2+P\right)_+^\lambda\right], \tag{45}$$

and the distribution $\delta^{(k)}(c^2 + P)$ is defined as follows:

$$\left\langle \delta^{(k)}\left(c^2+P\right), \varphi(x)\right\rangle = \left\langle \delta^{(k)}(s), \psi(s)\right\rangle = (-1)^k\psi^{(k)}(0). \tag{46}$$

According to (15), $\psi(s)$ is given by the formula

$$\psi(s) = \int_{c^2+P=s}\varphi(x)W_{c^2+P}(x,\mathrm{d}x). \tag{47}$$

The distribution $(c^2 + P)^{-k}$ can be defined, using (47), in the following way:

$$\begin{aligned} &\left\langle \left(c^2+P\right)^{-k}, \varphi(x)\right\rangle = \left\langle s^{-k}, \psi(s)\right\rangle \\ &\quad = \int_0^\infty s^{-k}\left[\psi(s)-\psi(0)-\cdots-\frac{s^{k-1}}{(k-1)!}\psi^{(k-1)}(0)H(1-s)\right]\mathrm{d}s. \end{aligned} \tag{48}$$

According to these definitions, we can simply substitute x for $c^2 + P$ in both sides of (35) to get

$$\left(c^2+P\right)^{-n}\delta^{(m-1)}\left(c^2+P\right) = \frac{(-1)^n}{2}\frac{(m-1)!}{(m+n-1)!}\delta^{(m+n-1)}\left(c^2+P\right). \tag{49}$$

4. A physical example

We give now a physical example. We consider a massless scalar $(\lambda/4!)\phi^4(x)$ theory in four dimensions. For this theory we shall evaluate the self-energy Green function.

The Lagrangian of this theory is

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{\lambda}{4!}\phi^4. \tag{50}$$

In the usual metric $(1, -1, -1, -1)$ the propagator for the field $\phi(x)$ is [13]

$$\Delta(x) = \left[-4\pi^2\left(u^2 - \mathrm{i}0\right)\right]^{-1}. \tag{51}$$

According to equations (A.1)–(A.6) of the appendix, we can write

$$\begin{aligned}\left(u^2 - \mathrm{i}0\right)^{-1} = {} & (2x_0)^{-1}\left[(x_0 - r)^{-1} + (x_0 + r)^{-1}\right] + (2r)^{-1}\left[\delta(x_0 - r) + \delta(x_0 + r)\right] \\ & + C\delta(x_0 - r)\delta(x_0 + r),\end{aligned} \tag{52}$$

where C is an arbitrary constant appearing in the definition of some distributions [13, 8.8, 8.9] (see also the appendix).

Using equation (37) and the relation given in [1], namely,

$$\frac{\delta(x_0 - r)\delta(x_0 + r)}{2r^2} = \frac{\pi}{2}\delta(x_0, x_1, x_2, x_3), \tag{53}$$

it is easy to show with the aid of (35) that

$$\left(u^2 - \mathrm{i}0\right)^{-1}\left(u^2 - \mathrm{i}0\right)^{-1} = \left(u^2 - \mathrm{i}0\right)^{-2}.$$

Then, we have for the self-energy:

$$\Sigma(x) = \left(\Delta(x)\right)^2 = \frac{1}{16\pi^4}\left(u^2 - \mathrm{i}0\right)^{-2}, \tag{54}$$

where $(u^2 - \mathrm{i}0)^{-2}$ is defined in [13, 8.8, 8.9] (see also the appendix).

5. Discussion

When we use the perturbative development in quantum field theory, we have to deal with products of distributions in configuration space, or else, with convolutions in the Fourier transformed p-space. Unfortunately, products or convolutions (of distributions) are in general ill-defined quantities. However, in physical applications one introduces some "regularization" scheme (for example, the dimensional regularization method [2,3,12]), which allows us to give sense to divergent integrals. A similar procedure to the use of "regulators" is the canonical product of Guelfand and Shilov.

In this work we have evaluated the canonical products $x^{-n} \cdot \delta^{(m-1)}(x)$, $x^{-n} \cdot x^{-m}$, $\delta^{(n)}(x) \cdot \delta^{(m)}(x)$, in the sense of Guelfand–Shilov and their generalization to the n-dimensional space. With the use of these products and the results of [1,4,5], we have showed that it is possible to obviate the use of any regularization method to calculate (directly) the product of propagators of particles.

Appendix. Definitions

From [13] we have

$$\delta^{(m)}\big(u^2\big) = \delta^{(m)}\big(x^0 + r\big)\big(x^0 - r\big)^{-m-1}\,\mathrm{sgn}\big(x^0 - r\big) + \delta^{(m)}\big(x^0 - r\big)\big(x^0 + r\big)^{-m-1}\,\mathrm{sgn}\big(x^0 + r\big), \tag{A.1}$$

where

$$u^2 = x_0^2 - x_1^2 - \cdots - x_{n-1}^2, \tag{A.2}$$

$$r^2 = x_1^2 + x_2^2 + \cdots + x_{n-1}^2, \tag{A.3}$$

$$\big(u^2 \pm \mathrm{i}0\big)^{-m} = u^{-2m} \pm \frac{(-1)^m}{(m-1)!}\,\mathrm{i}\pi\delta^{(m-1)}\big(u^2\big), \tag{A.4}$$

$$x^{-m}\,\mathrm{sgn}(x) = \frac{(-1)^{m-1}}{(m-1)!}\big\{|x|^{-1}\big\}^{(m-1)}, \tag{A.5}$$

$$|x|^{-1} = \big\{\mathrm{sgn}(x)\ln|x|\big\}' + C\delta(x), \tag{A.6}$$

where C is an arbitrary constant.

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